MULTIPRODUCT PRODUCTION SCHEDULING FOR STYLE GOODS WITH LIMITED CAPACITY, FORECAST REVISIONS AND TERMINAL DELIVERY*

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April 1971

522-71

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*Paper presented at The XVII International Conference of the Institute of Management Sciences, Imperial College, London, England, July 1-3, 1970. The authors gratefully acknowledge helpful comments by Joseph Thomas and Al Michel of Cornell University. Rein Peterson was supported in this work by the Faculty Research Fund of the Columbia University Graduate School of Business and the Associates Fund at the University of Western Ontario.

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ABSTRACT

A general mathematical model is formulated for the problem of scheduling production quantities for a group of products with seasonal stochastic demand through a common production facility. It is assumed that revised forecasts of total demand over the season for each product become available as the season progresses; delivery is not required until the end of the selling season. Limited production capacity requires that some production take place early in the season, when forecasts are less accurate. At the end of the season, there are overage costs and underage costs representing costs of producing excess quantities and opportunity costs of not producing enough units. Under certain assumptions concerning the data-generating process for forecast revisions, it is possible to formulate the entire problem as a dynamic programming problem; however, the formulation is not computationally feasible if two or more products are considered. Three heuristic approaches to the multiproduct problem are presented and their cost performance is evaluated in some numerical examples. In these particular examples, more frequent reforecasting and rescheduling produces substantial reduction in costs.

## I. INTRODUCTION

Consider a production scheduling problem in which M products are produced on a single common production facility and orders are obtained over a relatively short selling season. Delivery is not required until the end of the selling season, linear production costs are present, and production capacity is limited. Let the total time period up to the delivery date be divided up into a number of production periods, and assume that a revised forecast of total demand for each product is available at the beginning of each period. Under these circumstances, how should production be scheduled each period so as to minimize total costs, including costs of overage and underage at the end of the season? In this paper we develop a dynamic programming formulation for this general problem, point out its practical limitations and consider some alternative heuristic approaches.

This general problem was abstracted from an actual production sheduling problem faced by a garment manufacturing firm producing style goods. Similarly, wholesalers' orders for the December (Christmas) selling season may create a similar type of production scheduling problem for a wide variety of products.

A number of authors have explored the problem of allocating limited capacity in a nonseasonal situation (see [2], [5] and [16]), making use of (respectively) the Lagrange multiplier technique, quadratic programming, or adjustment of reorder points. In the seasonal setting, Spurrell [14] presents a plan for capacity allocation which is a special case of the multi-product classical newsboy problem with a Lagrange multiplier used to allocate limited capacity. However, Spurrell's approach makes no provision for forecast revisions. Chang and Fyffe [3] present a method to generate forecast

revisions as demands occur; their method continues to place some weight on the initial forecast, thus giving it a Bayesian characteristic. They do not concern themselves with the associated production scheduling decision problems, however. Hertz and Shaffir [8] present a forecasting procedure involving a simple extrapolation of cumulative wholesale demands through the season. For production allocation purposes they recommend a continuing newsboy approach using the most recent forecasts and associated forecast errors; they do not consider limited capacity.

Liff [11] and Wolfe [17] have found that retail sales of style goods tend to be proportional to inventory displayed, thereby creating an exponential distribution (assuming no reorders) of sales through the season. Their work seems useful in making forecast revisions for sales at the retail level, as compared to simple extrapolation using the Normal-shaped cumulative sales curve for wholesale demands (see [8] and [10]).

None of the work cited so far has contained both an explicit probabilistic model for forecast revisions and an optimal decision framework for such a model. Wadsworth [15] presents a 2-period example for a production scheduling problem for one product with limited capacity and with a revised (perfect) forecast available in the second period. Murray and Silver [12] present a Bayesian model for forecast revision (based on recent sales) imbedded in a sequential decision framework for which dynamic programming provides an optimal solution. Their work involves only one product, and the present study may be viewed as an extension of their work to the multiproduct case where the question of limited production capacity has greater complexity. Also, as will be seen subsequently, the forecasting mechanism allowed in the

present study may be a human one which is Bayesian in a broad sense.

Finally, Evans [4] has considered a problem similar to the one considered here except that in his model, demands for each product in a particular period are given by a random vector which is independent from one period to the next. Thus in his case, there is no forecast revision process involved. Another difference is that delivery is period by period rather than terminal. Evans demonstrates that under his cost assumptions, there exists a single (multidimensional) critical point S such that if initial inventory and production capacity allow, the optimal policy is to "order-up-to" S (in a multidimensional sense). If capacity restrictions are active, then the optimal policy is "order-up-to $Z^0(X)$" where X is a vector representing initial inventory and $Z^0(X)$ is a vector representing an optimal "order-up-to" multidimensional point given the production restriction. If any product were in excess supply it would not be produced. Evans concludes that "... The $Z^0$ function does not appear to have any particularly simple form."[1]

The problem under consideration here appears to be even more complex than Evans' since we allow for a stochastic forecast revision process, thereby nullifying the period-to-period independence of the random demand variable. Thus it is not surprising that an optimization approach to the problem becomes intractable for more than one product, and heuristic approaches must be used.

## II. A MODEL OF THE FORECAST DATA-GENERATING PROCESS

It has been shown elsewhere[2] that under certain circumstances, systems

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[1] See reference [4], p. 183.

[2] See reference [7].

for generating and revising forecasts of some unknown quantity may have the Markovian property of no memory. Specifically, in a number of cases it was found that ratios of successive forecasts could, as a first approximation, be treated as independent random variables distributed according to the Lognormal distribution. It should be noted that, for our purposes, the precise functional form of the probability density function relating successive forecasts is unimportant, as long as some approximate probability density function can be obtained. However, the assumption of independence is crucial to our dynamic programming formulation, since the current forecast is a sufficient statistic only if independence exists.

Based on the results reported in [7], it is assumed subsequently that a significant class of forecasting mechanisms exhibits the Markovian property. Specifically, in the remainder of this paper it is assumed that ratios of successive forecasts of total orders for a seasonal product are mutually independent Lognormal variates whose parameters may be estimated by analysis of historical forecast data. Let $X_j$ represent the forecast of total seasonal demand for a product, with the forecast made at the beginning of period $j$; then ratios of successive forecasts are $(X_{j+1}/X_j)$. Also let $X_{N+1}$ represent actual total demand for the product. Define

(1) $$Z_j = (X_{j+1}/X_j), \qquad j = 1,\dots, N \text{ (time periods)}$$

Then it is assumed that the variable $Z_j$ has the following distribution:

$$Z_j \sim f_{LN}(Z_j|\mu_j, \sigma_j) = \frac{1}{\sqrt{2\pi}\sigma_j Z_j} e^{-(\log Z_j - \mu_j)^2/2\sigma_j^2}, \qquad j = 1,\ldots N,$$

with $Z_j$ independent of $Z_k$ for $j \neq k$.

## III. DYNAMIC PROGRAMMING FORMULATION OF SINGLE-PRODUCT CASE

The general production scheduling problem described above is first formulated for the case of a single product. While this case is much less interesting than the multiproduct case, it still presents a non-trivial problem.

The state variable at the beginning of any period j is defined as a 2-dimensional vector with its first element being the current (revised) forecast $X_j$ of total demand and its second element being the current amount of inventory (call it $Y_j$) of the product produced in previous periods 1, 2, ...,j-1. That is, $Y_j$ represents the initial inventory at the beginning of period j. Decisions concerning the amount of product to produce in the $j^{th}$ period (for j = 1 to N only) are made at the beginning of the jth period and they are framed in terms of an "order-up-to" amount $Y_{j+1}$ of desired ending inventory (recall that no shipments will occur until the end of the season, in period N + 1). The amount to be produced in period j is $(Y_{j+1} - Y_j)$ for $j = 1,\ldots,N$. Also let:

$C_o$ = unit cost of overage in period N + 1 (i.e., the cost of having one unit of excess ending inventory after actual demand is known in period N + 1)

$C_u$ = unit cost of underage in period N + 1 (i.e., the cost of a shortage of one unit after actual demand is known)

$K$ = available production capacity[3] in each production period ($j = 1, \ldots N$).

Note that there are no setup costs, holding costs, or additional costs of production in this formulation; all costs are incurred in period $N + 1$, after production has ceased. (The terms $C_o$ and $C_u$ can, however, include a linear production cost.) Without loss of generality, a "unit" of product is defined to be the amount of product which can be produced with one unit of production capacity, with overage and underage costs, inventories and forecasts appropriately rescaled.

Now define the usual return function in dynamic programming as follows:

$g_j(X_j, Y_j)$ = minimum expected cost incurred from the beginning of period $j$ through period $N + 1$, given that the current state vector is $(X_j, Y_j)$ and that optimal decisions are made from period $j$ through period $N$.

Then the following relationship holds in period $N + 1$:

$$(2) \quad g_{N+1}(X_{N+1}, Y_{N+1}) = \begin{cases} C_o(Y_{N+1} - X_{N+1}) & \text{if } Y_{N+1} > X_{N+1} \\ C_u(X_{N+1} - Y_{N+1}) & \text{if } Y_{N+1} \leq X_{N+1} \end{cases}$$

This equation indicates that the costs in period $N + 1$ are simply the costs of overage or underage, whichever has occurred. (Recall that the actual demand has been written as a final "perfect" forecast $X_{N+1}$, consistent with reference [7].)

Working backward, consider the situation at the beginning of period $N$:

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[3] Any of the subsequent models could trivially be adapted to a different production capacity $K_j$ for each period $j$; also, production periods need not be of equal length.

$$(3) \quad g_N(X_N,Y_N) = \underset{Y_N \leq Y_{N+1} \leq Y_N + K}{\text{Min}} \left\{ \int_0^\infty g_{N+1}(X_N \cdot Z_N, Y_{N+1}) \cdot f_{LN}(Z_N | \mu_N, \sigma_N)\, dZ_N \right\}.$$

Equation (3) indicates that the optimal (minimum) expected cost from period N to the end of the process is obtained by minimizing the expected cost which will be incurred in period N + 1. The expression in curly brackets is the expectation of $g_{N+1}(X_N \cdot Z_N, Y_{N+1})$ with respect to the random variable $Z_N$; and the arguments of the function $g_{N+1}$ are the revised forecast next period, $X_{N+1} = X_N \cdot Z_N$, and the revised amount of inventory available at the beginning of the next period, $Y_{N+1}$. The minimization is performed in general[4] by allowing the decision variable $Y_{N+1}$ (the number of units "ordered-up-to" in period N) to vary over the allowable range from $Y_N$ (zero production) to $Y_N$ + K (100% use of capacity in period N), and selecting the value of $Y_{N+1}$ which results in a global minimization. Denote the optimal order-up-to value by $Y^*_{N+1}(X_N, Y_N)$ since the optimal decision will in general be a function of the current state vector. Then this value may be substituted into equation (3) to obtain a value for the return function $g_N(X_N, Y_N)$. The minimization process must be performed for every allowable state vector to generate a complete decision table $Y^*_{N+1}(X_N, Y_N)$ over all possible states.

Once the above operation has been completed for period N, we again work backward and repeat the process for period N-1, N-2, et cetera. The general recurrence relation for period j would be written as:

$$(4) \quad g_j(X_j,Y_j) = \underset{Y_j \leq Y_{j+1} \leq Y_j + K}{\text{Min}} \left\{ \int_0^\infty g_{j+1}(X_j \cdot Z_j, Y_{j+1}) \cdot f_{LN}(Z_j | \mu_j, \sigma_j)\, dZ_j \right\}$$

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[4] As will be indicated, the minimization of 3 may be performed analytically for stage N; however, for earlier periods it must be obtained by a search procedure.

Eventually, after working back to period 1, the return function $g_1(X_1, Y_1 = 0)$ would represent the minimum expected cost of initiating the process with zero inventory and an initial forecast of $X_1$ units. The optimal production decisions would be found at the beginning of each period by substituting the current forecast $X_j$ and the current inventory $Y_j$ into the appropriate optimal decision table $Y^*_{j+1}(X_j, Y_j)$.

Dynamic Programming Solution to Single-Product Case

Now the dynamic programming formulation above will be used to demonstrate that no general analytical solution exists even for the single-product case. Substituting equation (2) into equation (3), we obtain:

(5)

$$g_N(X_N, Y_N) = \min_{Y_N \le Y_{N+1} \le Y_N + K} \{C_o \int_0^{Y_{N+1}/X_N} (Y_{N+1} - X_N \cdot Z_N) \cdot f_{LN}(Z_N | \mu_N, \sigma_N) dZ_N$$

$$+ C_u \int_{Y_{N+1}/X_N}^{\infty} (X_N \cdot Z_N - Y_{N+1}) f_{LN}(Z_N | \mu_N, \sigma_N) dZ_N\}.$$

If $Z_N$ is distributed Lognormally with parameters $\mu_N$ and $\sigma_N$, it can be shown[5] that $X_{N+1} = X_N \cdot Z_N$ is distributed Lognormally with parameters $(\mu_N + \log_e X_N)$ and $\sigma_N$. Then (5) can be rewritten as

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[5]By definition, $\log_e X_N \sim N(\mu_N, \sigma_N)$. Now $\log_e X_{N+1} = \log_e(X_N \cdot Z_N) = \log_e X_N + \log_e Z_N$, with $\log_e X_N$ being a known constant at period N. Thus $\log_e X_{N+1} \sim N(\mu_N + \log_e X_N, \sigma_N)$ so that $X_{N+1} \sim LN(\mu_N + \log_e X_N, \sigma_N)$.

(6)

$$g_N(X_N,Y_N) = \underset{Y_N \leq Y_{N+1} + K}{\text{Min}} \{C_o \int_0^{Y_{N+1}} (Y_{N+1}-X_{N+1}) f_{LN}(X_{N+1}|\mu_N+\log_e X_N, \sigma_N) dX_{N+1}$$

$$+ C_u \int_{Y_{N+1}}^{\infty} (X_{N+1}-Y_{N+1}) f_{LN}(X_{N+1}|\mu_N + \log_e X_N, \sigma_N) d X_{N+1}\}.$$

Apart from the inequality restrictions on $Y_{N+1}$, the terms in brackets {} represent the usual overage and underage costs in a standard "newsboy" or "Christmas-tree" problem, in which the first-order condition for a minimum is as follows:

(7) $$F_{LN}(Y^*_{N+1}|\mu_N + \log_e X_N, \sigma_N) = \frac{C_u}{C_o + C_u}$$

Equation (7) indicates that the optimal amount of inventory $Y^*_{N+1}$ to "order up to", in the absence of any constraint, would be the value which satisfies the relationship indicated. Now the cumulative Lognormal distribution $F_{LN}(\cdot)$, just as the cumulative Normal distribution, cannot be expressed in closed form, and table look-up must be used to find the value of $Y_{N+1}$ which satisfies equation (7). Once that value is obtained, the constraint on the size of $Y_{N+1}$ can be taken into consideration as follows. Let the value of $Y_{N+1}$ satisfying (7) be denoted by $Y'_{N+1}$. Then since equation (6) is convex in $Y_{N+1}$ for $C_o \geq 0$ and $C_u \geq 0$, the optimal value of $Y_{N+1}$ under the existing constraint is given as follows:

$$(8)\quad Y^*_{N+1} = \begin{cases} Y_N & \text{if } Y'_{N+1} - Y_N \\ Y'_{N+1} & \text{if } Y_N - Y'_{N+1} - Y_N + K \\ Y_N + K & \text{if } Y'_{N+1} - Y_N + K \end{cases}$$

Since the cumulative of the Lognormal (or Normal) distribution cannot be expressed in closed form, it is not possible to find the optimal decision $Y^*_{N+1}(X_N, Y_N)$ as a closed-form function of the state vector, and thus it is not possible to substitute into equation (6) the optimal decision function $Y^*_{N+1}$ in order to obtain a closed-form solution for the return function $g_N(X_N, Y_N)$. It is, however, possible to approximate the solution to (6) by creating a grid of possible (discrete) values for the two-dimensional state vector $(X_N, Y_N)$ and similarly approximating the continuous probability density function of the random variable $X_{N+1}$ by a probability mass function defined over a set of discrete values. Then for stage N, equations (7) and (8) provide a solution which performs the required minimization in (6) (earlier stages cannot be minimized in this manner; a grid search over allowable decision values must be used for the general $Y_{j+1}$ term). The optimal decision $Y^*_{N+1}(X_N, Y_N)$ would be a function of the state vector, and would be tabulated and stored in the computer. Then the optimal decision would be substituted into equation (6) to obtain a complete table for the return function $g_N(X_N, Y_N)$. Finally, the above process of minimization, recording, and substitution would be repeated for periods N-1, N-2,...,1 to obtain the entire optimal policy (in tabular form).

## IV. DYNAMIC PROGRAMMING FORMULATION OF MULTIPRODUCT CASE

The preceding dynamic programming formulation may easily be generalized (at least in theory) to encompass M products. Define the following column vectors:

$\underline{X}_j = (X_{1,j}, X_{2,j}, \ldots, X_{M,j})'$ a vector of current (revised) forecasts at the beginning of period j, with

$i = 1, \ldots, M$ products and

$j = 1, \ldots, N + 1$ periods

(the prime denotes a transpose)

$\underline{Y}_j = (Y_{1,j}, Y_{2,j}, \ldots, Y_{M,j})'$, a vector of current inventories at the beginning of period j

$\underline{Z}_j = (Z_{1,j}, Z_{2,j}, \ldots, Z_{M,j})'$, a random vector of ratios of successive forecasts: $Z_{i,j} = X_{i,j+1}/X_{i,j}$ for all i.

$\underline{Y}_{j+1} = (Y_{1,j+1}, Y_{2,j+1}, \ldots, Y_{M,j+1})'$, a vector of desired ending inventories after period j (the decision vector).

Also let

$C_{o_i}$ = unit overage cost for product i ; $\underline{C}_o = (C_{o_1}, \ldots, C_{o_M})'$

$C_{u_i}$ = unit underage cost for product i ; $\underline{C}_u = (C_{u_1}, \ldots, C_{u_M})'$

K = available production capacity in each period.

Under the assumptions discussed above for forecast revisions, the $M \times 1$ random vector $\underline{Z}_j$ is a multivariate Lognormal variate with $\underline{Z}_j$ independent of $\underline{Z}_k$ if $j \neq k$:

$$\underline{Z}_j \sim f_{LN}^{(M)} \; (\underline{Z}_j | \underline{\mu}_j, \underline{\underline{V}}_j)$$

with

$$\underline{\mu} = (\mu_{1,j}, \mu_{2,j}, \ldots, \mu_{M,j})'$$

and

$$\underline{\underline{V}} = \begin{bmatrix} \sigma_{1,j}^2 & \sigma_{1,j;2,j} & \cdots & \\ \sigma_{2,j;1,j} & \sigma_{2,j}^2 & \ddots & \\ \vdots & & & \sigma_{M,j}^2 \end{bmatrix}$$

The state vector at the beginning of period j for the multiproduct case is the $2M \times 1$ vector $(\underline{X}_j', \underline{Y}_j')$; the decision vector is $\underline{Y}_{j+1}'$; and the return function will be denoted by $h_j(\underline{X}_j', \underline{Y}_j')$. If we define a "star" product $\underline{a} * \underline{b}$ of two vectors $\underline{a}$ and $\underline{b}$ by its $i^{th}$ element equalling the product $a_i \cdot b_i$, then the multiproduct analogue of equation (4) is:

(9)

$$h_j(\underline{X}_j', \underline{Y}_j') = \underset{\substack{0 \leq \underline{1}'(\underline{Y}_{j+1} - \underline{Y}_j) \leq K, \\ \underline{Y}_{j+1} \geq \underline{Y}_j}}{\text{Min}} \left\{ \int_{\underline{0}}^{\underline{\infty}} h_{j+1}((\underline{X}_j * \underline{Z}_j)', \underline{Y}'_{j+1}) \cdot f_{LN}^{(M)} \; (\underline{Z}_j | \underline{\mu}_j, \underline{\underline{V}}_j) d\underline{Z}_j \right\}, \; j=1,\ldots$$

with ending condition

(10)

$$h_{N+1}(\underline{X}'_{N+1},\underline{Y}'_{N+1}) = \sum_{i \ni Y_{i,N+1} > X_{i,N+1}} C_{o_i} (Y_{i,N+1} - X_{i,N+1}) + \sum_{i \ni Y_{i,N+1} \leq X_{i,N+1}} C_{u_i} (X_{i,N+1} - Y_{i,N+1}),$$

where $\underline{1}' = (1, 1,...,1)$, a $1 \times M$ row vector with all elements unity.

In theory, the set of equations (9) could be solved in the same manner as equation (4), except for two problems. Ignoring the manner in which the optimization of equation (9) subject to the inequality constraints would be performed, there is a much more serious difficulty in this equation. The dimensionality of the state space is 2M (two variables per product for M products), and numerical solutions to dynamic programming problems involving a dimensionality of the state space greater than three quickly become computationally infeasible,[6] assuming no special structure of the return function exists. In our case the return function involves a multivariate integration, which hardly qualifies as a simplifying special structure. Therefore the

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[6]See reference [6], p. 426.

preceding dynamic programming formulation serves only as a vehicle for problem description and is not useful as a framework for computational solution.

It should be noted, however, that the optimal value for the last decision in the multiproduct case (the choice of the vector $\underline{Y}_{j+1}$ in equation (9) for j = N) can be obtained through the use of Lagrange multipliers, as this problem is simply the constrained multiproduct single period newsboy problem. We will carry out this analysis here because all of our subsequent heuristics are derived from it.

After substituting (10) into (9) for j = N, form the function[7]

(11)

$$L(\underline{X}_N', \underline{Y}_N', \underline{Y}_{N+1}') = \sum_{i=1}^{M} C_{o_i} \int_0^{Y_{i,N+1}} (Y_{i,N+1} - X_{i,N+1}) f_{LN_i}(X_{i,N+1})\, d\, X_{i,N+1}$$

$$+ C_{u_i} \int_{Y_{i,N+1}}^{\infty} (X_{i,N+1} - Y_{i,N+1}) f_{LN_i}(X_{i,N+1})\, d\, X_{i,N+1} \qquad + \lambda[\sum_{i=1}^{M} (Y_{i,N+1} - Y_{i,N}) - K]$$

where $f_{LN_i}(X_{i,N+1})$ is the marginal probability density function of $X_{i,N+1}$. Taking partial derivatives of L with respect to each element of the decision vector and with respect to $\lambda$ produces:

(12)
$$\frac{\partial L}{\partial Y_{i,N+1}} = C_{o_i} F_{LN_i}(Y_{i,N+1}) - C_{u_i} [1 - F_{LN_i}(Y_{i,N+1})] + \lambda = 0$$

for i = 1,...,M and

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[7] It can be shown that even if independence across products is not present, equation (11) appropriately represents expected costs, since the cost function across all M products is separable.

(13) $$\frac{\partial L}{\partial \lambda} = \sum_{i=1}^{M} (Y_{i,N+1} - Y_{i,N}) - K = 0.$$

From (12) one can derive the following set of equations representing the solution to a constrained multiproduct newsboy problem:

(14) $$F_{LN_i}(Y_{i,N+1}) = \frac{C_{u_i} - \lambda}{C_{o_i} + C_{u_i}}, \quad i = 1,\ldots,M, \text{ where } F_{LN_i}(Y_{i,N+1}) =$$

$$\int_0^{Y_{i,N+1}} f_{LN_i}(X_{i,N+1})dX_{i,N+1}$$

where the parameters of $F_{LN_i}(\cdot)$ are $(\mu_{i,N} + \log_e X_{i,N}, \sigma_{i,N})$ following footnote 5. Equation (14) must be solved (for each i) for various trial values of $\lambda$ until the solutions $\{Y_{i,N+1}\}$ satisfy (13). However, the behavior of (13) is obviously monotonic in $\lambda$; as $\lambda$ is increased, reflecting a higher premium cost or shadow price on capacity in the last period, the "order-up-to" quantities $\{Y_{i,N+1}\}$ satisfying (14) will decrease, as will their sum. Thus, the search procedure is not difficult, and the $\{Y_{i,N+1}\}$ quantities so obtained will be optimal decision values. If any of these optimal values is less than current inventory then, based on the convexity cited earlier, production for that product should be fixed at zero and the trial-and-error process of varying $\lambda$ should be repeated. Also, if $\lambda$ is negative then it should be set at zero, since in that case capacity is not a binding

constraint.[8] Unfortunately, it is not possible to solve previous-period decisions optimally, for the reasons cited earlier.

## V. THREE HEURISTIC APPROACHES TO THE MULTIPRODUCT CASE

### Introduction

In the multiproduct case the optimal decision in the last period is given by (14), that is, the constrained newsboy decision rule. Optimal decisions for earlier periods are computationally infeasible, however. Three heuristics are now considered, each of which is a variation on the basic constrained newsboy theme. From this point of view, the heuristics extend previous work done on related problems ([4], [8], [14], [15]).

### Heuristic 1 (H-1)

The mathematical intractability can be overcome easily if one is willing for the moment to ignore the future production opportunities (beyond the current period) and the corresponding forecast revisions inherent in the problem. Under such a myopic approach the decision-maker would make his production decisions at each period, j, *as if* the current period were the last available production period, using the constrained newsboy decision-rule given previously in (14). This heuristic, then, allocates the next K units of production capacity to their "best" use in the restricted sense

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[8] If appropriate, one could also formulate this multiproduct single-period newsboy problem with more than one capacity constraint to reflect a more complex production technology with multiple resource requirements and constraints. See Hodges and Moore [9] for an example of such a formulation, and Ziemba [18] for a careful discussion of numerical solution procedures.

that if the plant (but not the inventory) were to burn to the ground after the current period, expected profits would be the maximum over all other allocations.

Specifically, heuristic H-1 proceeds as follows. Let the order quantities at each period, j, be represented by the difference between an optimal target order-up-to level $Y^*_{i,N+1}$ and the current cumulative inventory level $Y_{i,j}$. The M order-up-to quantities $\{Y^*_{i,N+1}\}$ are obtained by solving the following set of M+1 equations:

$$(15)\quad F_{i,N-j+1}\,(Y^*_{i,N+1}) = \frac{C_{ui} - \lambda_1}{C_{ui} + C_{oi}}\ ,\ ;\ i = 1,2,\ldots,M$$

$$(16)\quad [\sum_{i=1}^{M} (Y^*_{i,N+1} - Y_{i,j}) - K\ ] = 0,$$

where $F_{i,N-j+1}\,(\cdot)$ is the cumulative distribution function[9] of actual demand for product i as forecasted in period j, $X_{i,N+1}$.

Heuristic 2 (H-2)

An alternative to H-1 would be a heuristic that attempts to obtain some of the benefit of the optimal (but computationally infeasible) decision policy by "looking ahead", even if in a relatively crude manner, to allow

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[9] Assuming $Z_{i,j}$ are independent Lognormal variates across time periods, the cumulative marginal distribution function $F_{i,N-j+1}\,(\cdot)$ for product i is cumulative Lognormal with parameters

$$\mu = \log_e X_j + \sum_{k=j}^{N} \mu_k,\ \sigma = \sqrt{\sum_{k=j}^{N} \sigma_k^2}$$ (see reference [1] and footnotes 5 and 7.)

for the future production opportunities and associated forecast revisions which will actually occur as the season progresses. H-2 and H-3 represent two ways in which this can be represented.

We define H-2 as follows. At each period, j, let the M target order-up-to decision quantities $\{Y^*_{i,N+1}\}$ be given by solving the following set of M + 1 equations:

$$(17)\quad F_{i,N-j+1}\,(Y^*_{i,N+1}) = \frac{C_{ui} - \lambda_2}{C_{ui} + C_{oi}} \qquad i = 1,2,\ldots,M$$

$$(18)\quad [\,\sum_{i=1}^{M} (Y^*_{i,N+1} - Y_{i,j}) - (N-j+1)\,K] = 0.$$

Using H-2 the decision-maker proceeds in the same way as under H-1, except that the target order-up-to quantities $Y^*_{i,N+1}$ are now determined on the basis of the total available production capacity in the remaining (N-j+1) periods (compare equations (18) and (16)). Of course, in any period j, there is only enough capacity to produce K units. The M actual production quantities (denoted by $Q_{i,j}$) for the current period are determined from the M target order-up-to quantities $Y^*_{i,N+1}$ by the following pro-rata scheme:

$$(19)\quad Q_{i,j} = [Y^*_{i,N+1} - Y_{i,j}] \,/\, (N-j+1)\ ;\ i = 1,\ldots,M.$$

Note that by means of (17), (18) and (19) a crude characteristic of "looking ahead" is introduced into the heuristic. By rationing capacity as in (17) and (18), the decision-maker first determines the appropriate mix

of products in his current production on the basis of the total remaining production capacity of $(N-j+1)K$. He then proceeds as if he is going to produce $(N-j+1)^{-1}$ of the total required production quantity of each product, i, in each of the next $(N-j+1)$ periods.

Heuristic 3 (H-3)

The third heuristic is identical to H-2 except in the method of pro-rata rationing of available capacity. Instead of (19), the current production quantities $Q_{i,j}$ are determined by:

$$(20)\quad Q_{i,j} = \left( \frac{[Y^*_{i,N+1} - Y_{i,j}]}{\sum_{i=1}^{M} [Y^*_{i,N+1} - Y_{i,j}]} \right) K \ , \quad i = 1,\ldots,M;$$

that is, the available capacity of the current period is fully allocated among the M products in proportion to currently perceived need among them. The only time that H-3 will not fully allocate the current production capacity is when the sum of the production requirement quantities $Q_{i,j}$ implied by (17) and (18) do not exceed K; then there is no need for any pro-rata calculation such as in (20). Thus the difference between H-2 and H-3 is that in early periods, H-3 will tend to allocate all current capacity in proportion to perceived needs, while H-2 will allocate a portion of current capacity based on a comparison of total perceived needs against total remaining capacity $(N-j+1)K$. In each of these two heuristics, however, the percentage allocation of actual production among products will be the same

in both cases, namely, that found by (17) and (18).

Note that when $j = N$ in the last decision period, H-1, H-2 and H-3 all reduce to the optimal constrained newsboy decision-rule given earlier in (14). The differences between the heuristics come from the manner in which capacity before the final period is allocated on the basis of current forecasts.

Advantages and Disadvantages of The Heuristics

All three of the heuristics capture the essential "newsboy" flavor of the problem for individual items, as they all react to individual $C_{o_i}$ and $C_{u_i}$ values. All three take into account the variability of actual demand $X_{i,N+1}$ about the current forecast $X_{i,j}$; and all use the Lagrange multiplier to ration scarce production capacity.

The disadvantages are as follows: the decision rules of H-1, H-2 and H-3 are optimal only when one of two unrealistic assumptions are made; either we act as though the only remaining capacity is that available in the current period (H-1), or we act as though no forecast revisions will be available between now and the end of the season (H-2 and H-3). Neither of these situations is true, thus making all three approaches heuristic rather than optimal. None of the heuristics takes explicit account of the availability of revised forecasts as time passes. None has the ability to take into account any differences in the pattern of uncertainty resolution[10] over time among products. Also, in cases where total capacity is large relative to

[10] E.g., delay production of type 1 product because the next revised forecast may be vastly improved, while producing type 2 product because the next revised forecast for it may not be very much "better" than the current one.

initial forecasts, heuristic H-1 (and to a lesser extent, H-3) will create an early build-up of inventory which may turn out to be excessive. Finally, it is worth emphasizing that these three heuristics are relatively complex and implementation would be non-trivial (a computer-based system would seem essential).

## VI. NUMERICAL RESULTS

The performance of the three heuristic approaches described above was simulated over 100 trials of an N-period sequential production scheduling problem with three products (M=3). The results of the simulations are presented in terms of the mean and standard deviation of overage and underage costs for 100 trials. Unfortunately, it is not possible to provide a "benchmark" in the form of the (minimum) cost of an optimal sequential allocation rule, since it was demonstrated above that such a rule is computationally infeasible for two or more products, Estimates of current management performance on our numerical example have not been sought because it was felt that actual managers facing related problems typically encounter many more than three products. Moreover, their particular internalized heuristics may react partly to elements of their actual problems which are absent from our numerical example (e.g., setup costs and times, intermediate delivery requirements, nonlinear cost structure, etc.). A definite lower bound on expected underage and overage costs is zero, but with any forecast error at all, this would seem not to be a very tight lower bound. Nevertheless, the relative performance of the three heuristics can be assessed against zero costs to gain an idea of the amount of improvement

one of them provides relative to the others. In general, though, this research must be viewed as exploratory, and the three heuristics tested can be compared rigorously only against one another.

There were 27 simulation runs, consisting of the three heuristics compared under three cost cases with N, the number of production periods, set at three levels: N=6, N=3, and N=1. That is, there were initially six production periods, each with capacity K = 50; subsequently they were grouped into three equal periods (N=3) with K = 100 (thereby using only forecasts 1, 3 and 5), and then into one production period with K = 300 for the entire season.[11] For this last version the three heuristic policies are identical and "optimal" for the one-period problem presented.

The initial forecasts for the three products were $\underline{X}_1 = (33,67,100)'$. The forecast revision process was assumed to be as stated earlier [see equation (1)], with ratios of successive forecasts being mutually independent Lognormal variates. For our numerical example we also assumed independence among the forecast revisions for the three products at a given period; i.e., it was assumed that $Z_{1,j}$, $Z_{2,j}$ and $Z_{3,j}$ [see above equation (9)] were independent[12]

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[11] See footnote 9 for the derivation of appropriate $(\mu,\sigma)$ parameters for these N = 3, N = 1 cases.

[12] While this assumption could be criticized in some practical situations, we feared that any alternative assumption could seriously bias our numerical results.

We also set parameters[13] $\mu_{i,j} = 0$, and set $\sigma_{ij}$ as follows (initially, the same for all i):

| Period j: | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $\sigma_{ij}$ : | 0.18 | 0.15 | 0.12 | 0.09 | 0.06 | 0.03 |

The three cost cases were as follows: Case One had $C_{u_i} = 1$ and $C_{o_i} = 2$ for all i, with $\sigma_{ij}$'s as specified above. Case Two had the same $C_{u_i}$ and $C_{o_i}$ values, but this time the values $\sigma^2_{3,j}$ were exactly twice as large as $\sigma^2_{ij}$ implied by the initial specification for i = 1 and 2. Finally, Case Three involved the initial case of equal $\sigma_{ij}$ values for all three products but this time there were three separate values for overage and underage costs for each product:

$$\underline{C}_u = (1,1,2.33)' \text{ and } \underline{C}_o = (2,1,1)'$$

The results of the 27 simulations are contained in Table 1. Table 2 contains a representative set of frequency distributions for the data summarized in Table 1. Finally, Table 3 contains the results of one particular

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[13] It can be shown that $E[Z_j] = \exp(\mu_j + \sigma_j^2/2)$. For an unbiased forecast we require $E[Z_j] = 1$. Since $\sigma_j^2 > 0$, we could have set $\mu_j = -\sigma_j^2/2$ to obtain an unbiased forecasting process. Setting $\mu_{ij} = 0$ merely creates a small, systematic forecast bias the amount of which is precisely known and precisely accounted for; it was done for computational convenience.

random trial, illustrating how the three heuristics actually behaved in a particular situation.

## VII. SOME CONCLUSIONS

From Table 1, Heuristic H-2 always produces the lowest average cost, as compared to the other heuristics; however, heuristic H-3 generally produces cost outcomes with the least amount of variability (also see Table 2). Most of these differences are not statistically significant, however. Mean cost for heuristic H-1 significantly exceeds that of the other heuristics for the N=6 situation; apparently the very large values for $\lambda_1$ in early periods create an inventory imbalance which is not flexible enough to cope with subsequent forecast changes. Conversely, H-2 and H-3 contain simple pro-rata rules which tend to avoid extreme imbalance in the inventory mix. H-3 probably suffers from its tendency toward excessive "early" production as compared to H-2. On the other hand, the very large cost outcomes of H-2 (contributing to its larger variance relative to H-3) result from situations in which forecasts are revised upward in periods 3 and 4, when remaining capacity will not suffice to meet enlarged demand. A risk-averting decision-maker might decide to use heuristic H-3 (or some variant of it) in order to avoid the few really disastrous outcomes possible under H-2, while an expected-value minimizer would prefer H-2, based on our sample results. The existence of an inventory holding cost would also tend to favor H-2.

It is interesting to study the effect of a larger number of (smaller)

Table 1

Simulation Results (100 trials each): Costs of Underage and Overage

| Case | No. of Periods | H-1 Mean Cost | H-1 Standard Deviation† | H-2 Mean Cost | H-2 Standard Deviation | H-3 Mean Cost | H-3 Standard Deviation |
|---|---|---|---|---|---|---|---|
| **One**: $\underline{C}_u = (1,1,1)'$, $\underline{C}_o = (2,2,2)'$; all variances equal | N = 6 | 22.2 | 23.8 | 11.2* | 14.9 | 12.6 | 7.8* |
| | N = 3 | 23.8 | 15.8 | 19.5* | 20.9 | 23.6 | 15.5* |
| | N = 1 | 57.1 | 29.6 | ** | ** | ** | ** |
| **Two**: $\underline{C}_u = (1,1,1)'$, $\underline{C}_o = (2,2,2)'$; variance for j=3 is twice that of j=1 and j=2 | N = 6 | 27.7 | 36.9 | 17.2* | 27.1 | 18.5 | 19.4* |
| | N = 3 | 30.9 | 34.4 | 26.0* | 35.1 | 30.6 | 34.1* |
| | N = 1 | 69.7 | 43.5 | ** | ** | ** | ** |
| **Three**: $\underline{C}_u = (1,1,2.33)'$, $\underline{C}_o = (2,1,1)'$; all variances equal | N = 6 | 17.0 | 10.4 | 9.5* | 12.6 | 12.6 | 6.8* |
| | N = 3 | 25.1 | 17.5* | 19.2* | 26.4 | 24.2 | 21.5 |
| | N = 1 | 76.7 | 61.8 | ** | ** | ** | ** |

*Lowest value (of the three heuristics considered)

**Not run, would be identical to H-1 results.

†Of actual cost for the 100 trials. The appropriate standard derivation of mean cost would be $\sigma_{\bar{x}} = \sigma_x/\sqrt{n} = \sigma_x/10$.

Table 2

Representative Results In Frequency Distribution Form

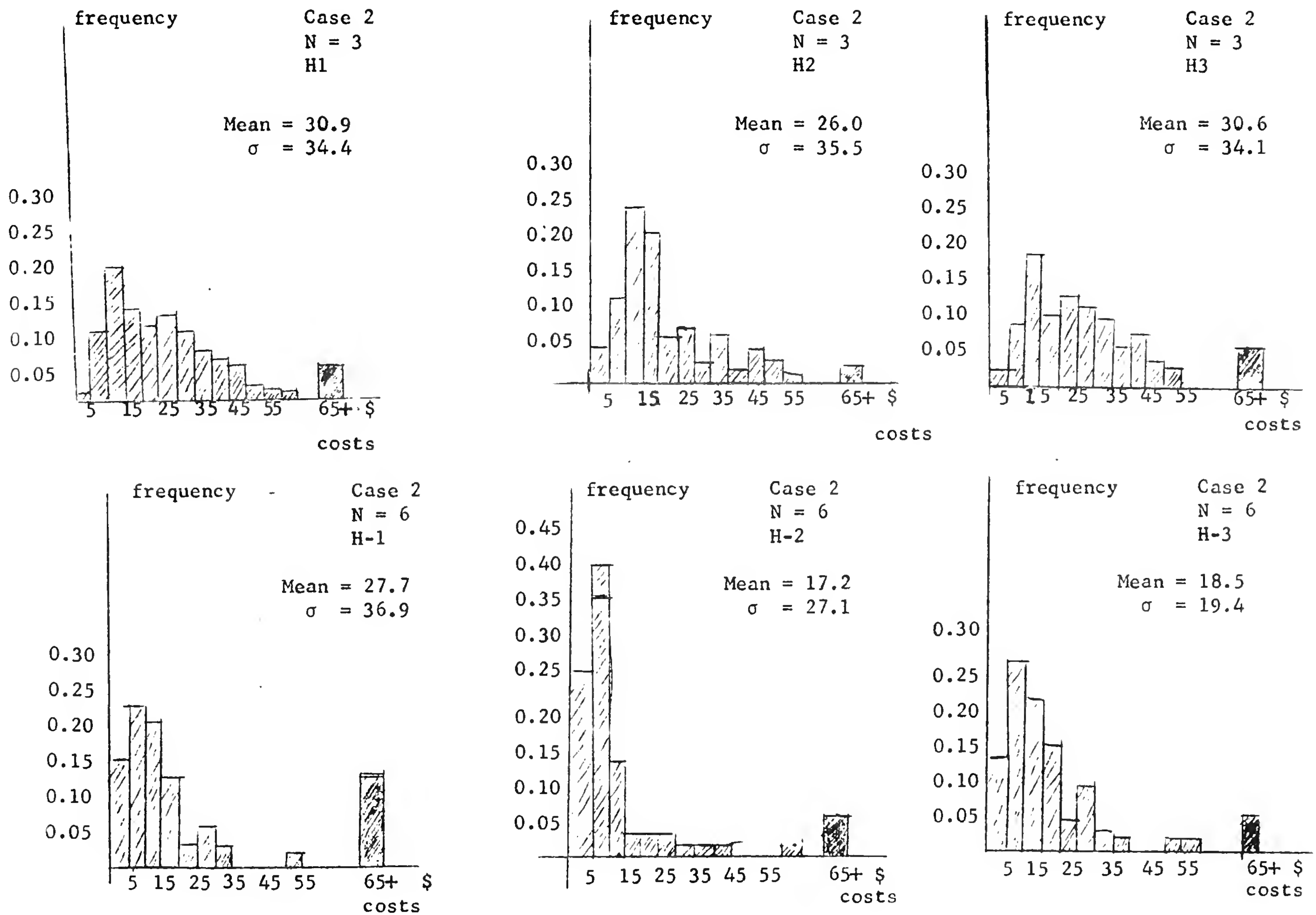

Table 3

Example of A Random Trial (Case 2; N=3)

| Period | Product | Forecast | H1 | | H2 | | H3 | |
|---|---|---|---|---|---|---|---|---|
| | | | Cum. Inventory | Production Decisions | Cum. Inventory | Production Decisions | Cum. Inventory | Production Decisions |
| 1 | 1 | 33 | 0. | 18 | 0. | 10 | 0. | 16 |
| | 2 | 67 | 0. | 38: | 0. | 20 | 0. | 34 |
| | 3 | 100 | 0. | 44 | 0. | 29 | 0. | 50 |
| 2 | 1 | 27 | 18 | 6 | 10 | 7 | 16 | 9 |
| | 2 | 65 | 38 | 23 | 20 | 20 | 34 | 26 |
| | 3 | 87 | 44 | 35 | 29 | 25 | 50 | 29 |
| 3 | 1 | 32 | 24 | 7 | 17 | 14 | 25 | 6 |
| | 2 | 45 | 61 | 0. | 40 | 4 | 60 | 0 |
| | 3 | 99 | 79 | 16 | 54 | 41 | 79 | 16 |
| Actual | 1 | 30 | 31 | - | 31 | - | 31 | - |
| | 2 | 50 | 61 | - | 44 | - | 60 | - |
| | 3 | 103 | 95 | - | 95 | - | 95 | - |
| Product Cost | 1 | | over | \$ 2.00 | over | \$ 2.00 | over | \$ 2.00 |
| | 2 | | over | \$22.00 | under | \$ 6.00 | over | \$20.00 |
| | 3 | | under | \$ 7.00 | under | \$ 7.00 | under | \$ 7.00 |
| Total Cost | | | | \$31.00 | | \$15.00 | | \$29.00 |

time

production periods and associated reforecasting and rescheduling. As the total season is divided from one entire production period, with no forecast revisions available (N = 1), to the case N = 3 with two forecast revisions and production rescheduling, the expected costs of H-2 and H-3 drop sharply, a typical cost reduction being 60%. Subsequent cost improvement from N = 3 to N = 6 provides less of a reduction in absolute terms; typically the reduction is about 60% of the cost of the N = 3 case. As more production subperiods are created and more forecast revisions are made, further improvements in costs would occur, but no doubt with diminishing returns.[14] These results indicate the importance of making at least heuristic adjustments based on revised forecasts, as opposed to freezing a production plan at the outset; they also indicate that at some point, the additional effort expended in providing more frequent forecast revisions (and associated production decisions) may exceed the benefits obtained.

Finally, although all of the conclusions about magnitudes of effects are based on the specific numerical examples described above, we conjecture that the general characteristics of this type of production scheduling problem have been captured by our numerical examples; thus the nature of the results should hold under a variety of similar problem settings. Nevertheless, it is recommended that decision-makers facing this type of production scheduling problem use the framework described herein to construct their own

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[14] Such returns are also dependent on the general "tightness" of the production scheduling problem, as indicated by the ratio of the sum of all initial forecasts $\sum_i X_{i,1}$ to total available capacity (NK).

simulations with their particular costs, capacity limits, and other parameters. The results of such simulations will provide more precise guidelines in choosing among these three heuristics (or others) and deciding on the most desirable value for N, the number of forecast revisions.

## Further Research

We have not exhausted all possibilities for reasonable heuristics to attack this production scheduling problem; other candidates undoubtedly exist. However, it seems reasonable to expect that better heuristics will be even more complex than those presented above. Moreover, the omission of setup costs and their corresponding impact on available productive capacity is a serious shortcoming of the formulation for many potential applications. Finally, in some situations the differences in uncertainty resolution over time among products may be substantial; if this is so, a good heuristic should try to take this phenomenon into account. From these considerations it seems apparent that the production scheduling problem considered in this paper is a fruitful one for further research.

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